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A sequential test with three possible decisions for comparing two unknown probabilities, based on groups of observations

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1. Introduction.

We consider two series of independent trials, e.g. two processes A and B, each trial resulting in a success or a failure with probabilities p, $1-p$ and p', $1-p'$ for the two processes respectively.

A sequential test with two possible decisions for the comparison of p and p', developed by WALD [6], may be used if the trials are executed in pairs, each pair consisting of one trial for each process.

For groups of trials of both processes a sequential test with two possible decisions for comparing p and p' has been described in [8]. This test is carried out as follows.

Suppose the group of trials, constituting the i-th stage of the test, consists of n_i trials for process A and m_i for process B. If the numbers of successes are $\underline{a}_i$ and $\underline{b}_i$ [1] respectively, $\underline{a}_i$ and $\underline{b}_i$ both possess a binomial probability distribution with parameters n_i, p and m_i, p' respectively.

The following transformation is then used: if $\underline{n}'$ possesses a binomial probability distribution with parameters n and p the random variable

$$(1.1)\qquad \begin{cases} y = 2\arcsin\sqrt{\dfrac{n'}{n}} & 0 < n' < n \\ y = \sqrt{\dfrac{2}{n}} & n' = 0 \\ y = \pi - \sqrt{\dfrac{2}{n}} & n' = n \end{cases}$$

is, for large n, approximately normally distributed with mean

$$(1.2)\qquad \mu = 2\arcsin\sqrt{p}$$

and variance

$$(1.3)\qquad \sigma^2 = \frac{1}{n} + \frac{1}{n^2}\,.$$ [2]

The transformation (1.1) is applied to $\underline{a}_i$ and $\underline{b}_i$; after this transformation, these random variables will be denoted by $\underline{u}_i$ and $\underline{v}_i$ respectively.

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1) Random variables are denoted by underlined symbols; the same symbols, not underlined, are used to denote values assumed by these random variables.

2) Tables of y (in radians) and σ^2 are given in [8] for $n = 10(1)50$ and $0 \leq n' \leq n$.

The sequential test of WALD [6] with two possible decisions for the mean of a normal distribution with known variance is then applied to

$$x_i = u_i - v_i \qquad (i = 1, 2, \ldots\ldots).$$

Both abovementioned tests are tests with two possible decisions, i.e. the tests result in one of the decisions $p > p'$ or $p < p'$.

The sequential test for comparing p and p' for the case of pairs of trials may be generalized to a test with three possible decisions, i.e. a test resulting in one of the decisions $p > p'$, $p < p'$ or $p \approx p'$, by means of a test developed by DE BOER [2]. This case will not be considered here.

In this paper the abovementioned test for comparing p and p' for groups of trials will be generalized by means of a sequential test for the mean of a normal distribution with known variance developed by SOBEL and WALD [7].

The sequential test of WALD with two and the test of SOBEL and WALD with three possible decisions for the mean of a normal distribution with known variance will be described first.

2. Sequential test for the mean of a normal distribution with known variance.

2.1. Two possible decisions.

For the case that the successive observations $x_1, x_2, \ldots\ldots$ are idependent observations of one random variable $\underline{x}$, possessing a normal probability distribution with mean μ and known variance σ^2, WALD's sequential test with two possible decisions for μ has been described in [6] (p. 117-124). This test will be described here for the case that the variance is not constant. This results in a small change in WALD's test; the proof of the validity of this test follows at once from WALD's own proofs.

For the test a value μ_0 of μ must be chosen, the two possible decisions being: $\mu < \mu_0$ and $\mu > \mu_0$, where we may substitute $\leq$ resp. $\geq$ for $<$ resp. $>$.

Furthermore two values μ_1 and μ_2 must be chosen with

$$\mu_1 < \mu_0 < \mu_2$$

such that the decision $\mu > \mu_0$ is considered an incorrect decision if $\mu \leq \mu_1$ and the decision $\mu < \mu_0$ is considered incorrect if $\mu \geq \mu_2$; for values of μ between μ_1 and μ_2 it is not important which decision is taken.

The concepts "correct" and "incorrect decision" are thus defined as follows:

Table I

Correct and incorrect decisions

value of μ	correct decision	incorrect decision
$\mu \leq \mu_1$	$\mu < \mu_0$	$\mu > \mu_0$
$\mu \geq \mu_2$	$\mu > \mu_0$	$\mu < \mu_0$
$\mu_1 < \mu < \mu_2$	$\mu < \mu_0$ and $\mu > \mu_0$	—

The interval (μ_1, μ_2) is called the indifference region. If:

α = the probability of acceptance of $\mu > \mu_0$ if $\mu = \mu_1$

β = the probability of acceptance of $\mu < \mu_0$ if $\mu = \mu_2$,

and if α and β are chosen both $< \frac{1}{2}$ the probability of an incorrect decision is $\leq \alpha$ for $\mu \leq \mu_1$ and $\leq \beta$ for $\mu \geq \mu_2$.

The value μ_0 is of no further importance for the performance of the test.

The test is carried out as follows (σ_i^2 is the known variance of x_i):

Additional observations are taken as long as:

$$\frac{\ln B}{\mu_2 - \mu_1} < \sum_{i=1}^{n} \frac{x_i - \frac{\mu_1 + \mu_2}{2}}{\sigma_i^2} < \frac{\ln A}{\mu_2 - \mu_1}, \tag{2.1}$$

where

$$A = \frac{1-\beta}{\alpha} > 1$$

$$B = \frac{\beta}{1-\alpha} < 1.$$

The test is terminated as soon as (2.1) does not hold and the decision $\mu > \mu_0$ is then taken if

$$\sum_{i=1}^{n} \frac{x_i - \frac{\mu_1 + \mu_2}{2}}{\sigma_i^2} \geq \frac{\ln A}{\mu_2 - \mu_1}$$

and the decision $\mu < \mu_0$ if

$$\sum_{i=1}^{n} \frac{x_i - \frac{\mu_1 + \mu_2}{2}}{\sigma_i^2} \leq \frac{\ln B}{\mu_2 - \mu_1}.$$

If the random variables x_i all have the same variance σ^2 the test may be carried out graphically, as indicated by WALD [6] (p. 118-121).

2.2. Three possible decisions.

The sequential test with three possible decisions for the mean μ of a normal distribution with known variance, developed by SOBEL and WALD has been described in [7] for the case that the variance of x_i is a constant. This restriction is again dropped here.

For the test two values μ_0 and μ_0' and four values μ_1, μ_2, μ_3 and μ_4 must be chosen such that

$$\mu_1 < \mu_0 < \mu_2 < \mu_3 < \mu_0' < \mu_4 ,$$

the three possible decisions being:

1. $\mu < \mu_0$
2. $\mu > \mu_0'$
3. $\mu_0 \leq \mu \leq \mu_0'$.

The intervals (μ_1, μ_2) and (μ_3, μ_4) are the indifference regions.

The concepts "correct" and "incorrect decision" are defined as follows:

Table II

Correct and incorrect decisions

value of μ	correct decision	incorrect decision
$\mu \leq \mu_1$	$\mu < \mu_0$	$\mu_0 \leq \mu \leq \mu_0'$ $\mu > \mu_0'$
$\mu_1 < \mu < \mu_2$	$\mu < \mu_0$ $\mu_0 \leq \mu \leq \mu_0'$	$\mu > \mu_0'$
$\mu_2 \leq \mu \leq \mu_3$	$\mu_0 \leq \mu \leq \mu_0'$	$\mu > \mu_0'$ $\mu < \mu_0$
$\mu_3 < \mu < \mu_4$	$\mu_0 \leq \mu \leq \mu_0'$ $\mu > \mu_0'$	$\mu < \mu_0$
$\mu \geq \mu_4$	$\mu > \mu_0'$	$\mu < \mu_0$ $\mu_0 \leq \mu \leq \mu_0'$

The values μ_0 and μ_0' are of no further importance for the performance of the test.

Suppose T is the sequential test of section 2.1 for testing $\mu = \mu_1$ against $\mu = \mu_2$, then this test leads to a decision as soon as

(2.2) $$\frac{\ln B}{\mu_2 - \mu_1} < \sum_{i=1}^{n} \frac{x_i - \frac{\mu_1 + \mu_2}{2}}{\sigma_i^2} < \frac{\ln A}{\mu_2 - \mu_1}$$

does not hold, where

$$A = \frac{1-\beta}{\alpha}\ , \qquad B = \frac{\beta}{1-\alpha}\ ,$$

α = the probability of accepting $\mu \geq \mu_0$ according to T if $\mu = \mu_1$,
β = the probability of accepting $\mu < \mu_0$ according to T if $\mu = \mu_2$.

Suppose furthermore that T' is the analogous sequential test for testing $\mu = \mu_3$ against $\mu = \mu_4$ then T' leads to a decision as soon as

(2.3) $$\frac{\ln B'}{\mu_4 - \mu_3} < \sum_{i=1}^{n} \frac{x_i - \frac{\mu_3 + \mu_4}{2}}{\sigma_i^2} < \frac{\ln A'}{\mu_4 - \mu_3}$$

does not hold, where

$$A' = \frac{1-\beta'}{\alpha'}\ , \qquad B' = \frac{\beta'}{1-\alpha'}\ .$$

α' = the probability of accepting $\mu > \mu_0'$ according to T' if $\mu = \mu_3$

β' = the probability of accepting $\mu \leq \mu_0'$ according to T' if $\mu = \mu_4$.

We introduce the following notation

(2.4) $$\begin{cases} a = \dfrac{\ln A}{\mu_2 - \mu_1} & a' = \dfrac{\ln A'}{\mu_4 - \mu_3} \\ b = \dfrac{\ln B}{\mu_2 - \mu_1} & b' = \dfrac{\ln B'}{\mu_4 - \mu_3} \\ \displaystyle\sum_{i=1}^{n} \frac{x_i - \frac{\mu_1 + \mu_2}{2}}{\sigma_i^2} = y_n & \displaystyle\sum_{i=1}^{n} \frac{x_i - \frac{\mu_3 + \mu_4}{2}}{\sigma_i^2} = y_n'\ , \end{cases}$$

then according to T a decisions is taken as soon as

(2.2a) $$b < y_n < a$$

does not hold and according to T' as soon as

(2.3a) $$b' < y_n' < a'$$

does not hold.

From (2.4) it follows that a and a' are positive, b and b' negative and

$$y_n > y_n' \tag{2.5}$$

If now the inequalities

$$\begin{cases} b \leq b' \\ a \leq a' \end{cases} \tag{2.6}$$

are fulfilled, it follows from (2.5) and (2.6) that T' cannot lead to the decision $\mu > \mu_0'$ before the decision $\mu \geq \mu_0$ has been found according to T and that T cannot give the decision $\mu < \mu_0$ before the decision $\mu \leq \mu_0'$ has been given by T'.

In that case only the following decision according to T and T' are possible:

1. T' gives the decision $\mu \leq \mu_0'$ and T gives (at the same or a later step) the decision $\mu < \mu_0$ or the decision $\mu \geq \mu_0$.
2. T gives the decision $\mu \geq \mu_0$ and T' gives (at the same or a later step) the decision $\mu \leq \mu_0'$ or the decision $\mu > \mu_0'$.

The sequential test with three possible decisions is then defined as follows:

Additional observations are taken as long as not both tests T and T' have given a decision. As soon as both tests are terminated a decision is taken according to the following rules:

1. $\mu < \mu_0$ if T' has given the decision $\mu \leq \mu_0'$ and T the decision $\mu < \mu_0$,
2. $\mu > \mu_0'$ if T has given the decision $\mu \geq \mu_0$ and T' the decision $\mu > \mu_0'$,
3. $\mu_0 \leq \mu \leq \mu_0'$ if T has given the decision $\mu \geq \mu_0$ and T' the decision $\mu \leq \mu_0'$.

If (2.6) does not hold there exists the possibility of accepting $\mu > \mu_0'$ according to T' and of afterwards accepting $\mu < \mu_0$ ($< \mu_0'$) according to T. This kind of contradictory result is excluded by (2.6).

If the random variables x_i all have the same variance σ^2 the test may be carried out graphically, cf. [7].

3. Sequential test with three possible decisions for the comparison of two probabilities.

On the basis of the test of section 2.2 a sequential test with three possible decisions for comparing two unknown probabilities p and p' may be developed as follows.

To the variables a_i and b_i (see section 1) one of the following transformations is applied

$$(3.1) \qquad y = 2\arcsin\sqrt{\frac{n'}{n}}$$ 3)

$$(3.2) \qquad \begin{cases} y' = 2\arcsin\sqrt{\frac{n'}{n}} & 0 < n' < n \\ y' = 2\arcsin\sqrt{\frac{1}{4n}} & n' = 0 \\ y' = \pi - 2\arcsin\sqrt{\frac{1}{4n}} & n' = n \end{cases}$$ 4)

$$(3.3) \qquad \begin{cases} y'' = 2\arcsin\sqrt{\frac{n'}{n}} & 0 < n' < n \\ y'' = \sqrt{\frac{2}{n}} & n' = 0 \\ y'' = \pi - \sqrt{\frac{2}{n}} & n' = n \end{cases}$$

where n' possesses a binomial probability distribution with parameters n and p.

The transformation (3.1) is introduced by FISHER [4], the transformation (3.2) by BARTLETT [1] and (3.3) is given in [8]. For further information about the transformations we refer to [3] (p. 395-416).

Denoting the variables a_i and b_i, after their transformation, by u_i and v_i the sequential test of section 2.2 is applied to the random variables

$$x_i = u_i - v_i \qquad (i = 1, 2, 3, \ldots\ldots)$$

which possess, for large n_i and m_i, approximately a normal probability distribution with mean

$$(3.4) \qquad \mu = 2\arcsin\sqrt{p} - 2\arcsin\sqrt{p'} = 2\arcsin\left(\sqrt{pq'} - \sqrt{p'q}\right) \qquad \begin{matrix} q = 1-p \\ q' = 1-p' \end{matrix}$$

and variance

$$(3.5) \qquad \sigma_i^2 = \frac{1}{n_i} + \frac{1}{n_i^2} + \frac{1}{m_i} + \frac{1}{m_i^2}\ .$$

Two values μ_0 and μ_0' and four values μ_1, μ_2, μ_3 and μ_4 of μ must be chosen, with:

$$(3.6) \qquad \mu_1 < \mu_0 < \mu_2 < \mu_3 < \mu_0' < \mu_4$$

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3) Tables of $y = 2\arcsin\sqrt{x}$ are given in [5] for $x =$ = 0,000(0,001)1,000, p. 70-71, with y in radians.

4) Tables of $y = 2\arcsin\sqrt{\frac{1}{4n}}$ and $y = \pi - 2\arcsin\sqrt{\frac{1}{4n}}$ are given in [3], p. 406 for n = 10(1)50.

and four values α, α', β and β' (all $< \frac{1}{2}$) with (see (2.6))

$$(3.7) \qquad \begin{cases} \dfrac{\ln B}{\mu_2 - \mu_1} \leq \dfrac{\ln B'}{\mu_4 - \mu_3} \\[2ex] \dfrac{\ln A}{\mu_2 - \mu_1} \leq \dfrac{\ln A'}{\mu_4 - \mu_3} , \end{cases}$$

where

$$A = \frac{1-\beta}{\alpha} \qquad\qquad B = \frac{\beta}{1-\alpha}$$

$$A' = \frac{1-\beta'}{\alpha'} \qquad\qquad B' = \frac{\beta'}{1-\alpha'} .$$

Having chosen these values the abovementioned test may be applied, leading to one of the decisions:

$$(3.8) \qquad \begin{aligned} &1.\ \mu < \mu_0 \\ &2.\ \mu > \mu_0' \\ &3.\ \mu_0 \leq \mu \leq \mu_0' . \end{aligned}$$

We shall translate these decisions in terms of p and p'.

Let

$$(3.9) \qquad \sqrt{pq'} - \sqrt{p'q} = \delta ,$$

then

$$(3.10) \qquad \mu = 2 \arcsin \delta \qquad\qquad |\delta| \leq 1 .$$

The functional relationship (3.9) between p and p' for given δ^2 consists (see fig. 1) of the arcs PQ and RS of the ellipse:

$$(3.11) \qquad p^2 + p'^2 - 2pp'(1-2\delta^2) - 2\delta^2(p+p') + \delta^4 = 0$$

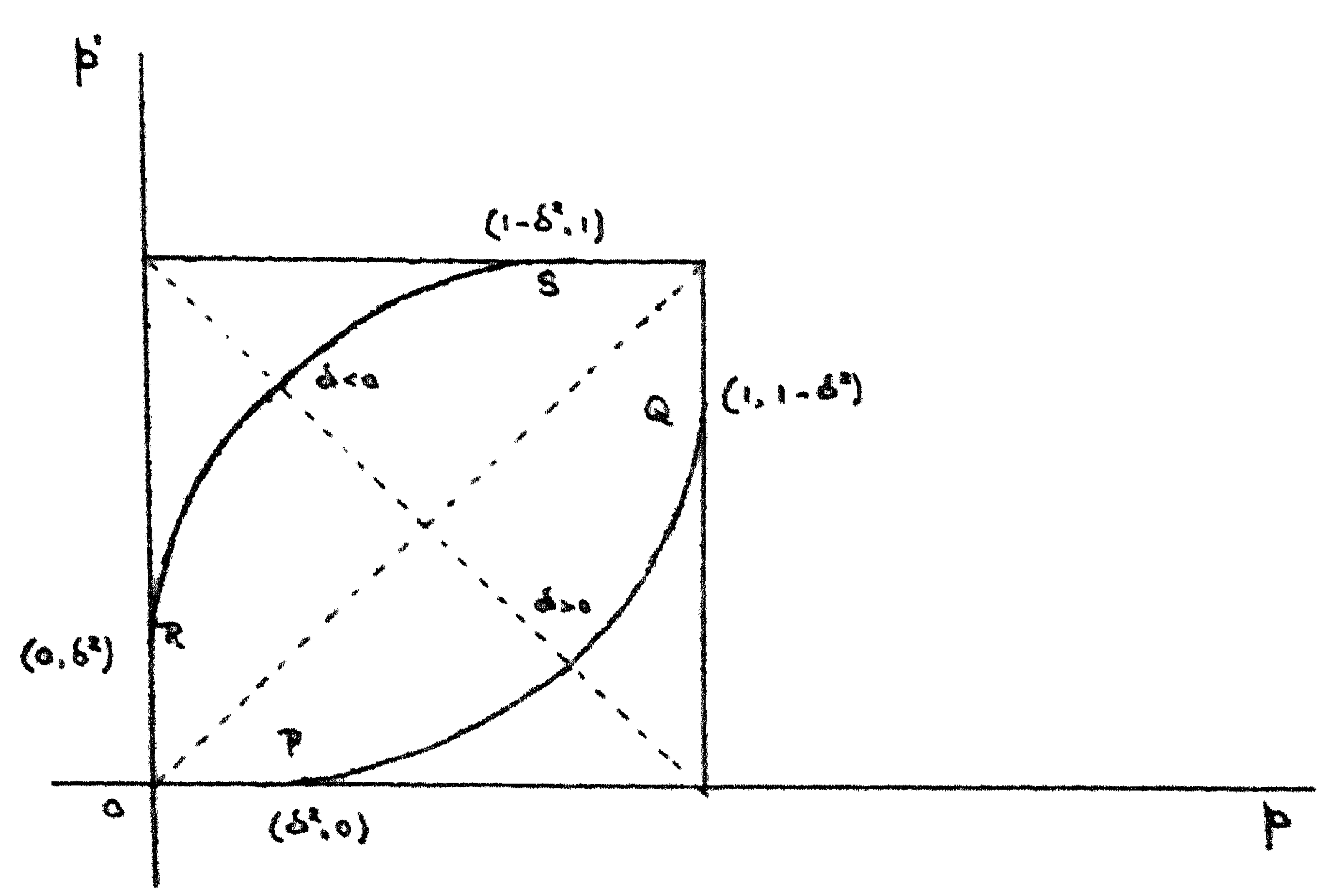


fig. 1.

Functional relationship between p and p' for given value of δ^2.

Choosing two values δ_0 and δ_0' and four values δ_1, δ_2, δ_3 and δ_4 of δ with

$$(3.12) \qquad \delta_1 < \delta_0 < \delta_2 < 0 < \delta_3 < \delta_0' < \delta_4 ,$$

the decisions (3.8) are equivalent with the following decisions for δ (see (3.10)):

(3.13)
$$1.\ \delta < \delta_0 \qquad 2.\ \delta > \delta_0' \qquad 3.\ \delta_0 \leq \delta \leq \delta_0'$$

and hence with the following decisions for p and p':

(3.14)
1. the point (p, p') lies above the arc RS of figure with $\delta = \delta_0$,
2. the point (p, p') lies below the arc PQ with $\delta = \delta_0'$,
3. the point (p, p') lies on or between the arcs PQ and RS with $\delta = \delta_0'$ and $\delta = \delta_0$ respectively.

The values δ_1, δ_2, δ_3 and δ_4 may be chosen by means of fig. 2, where the arcs PQ and RS are given for several values of δ.

One may also choose these values as follows:

1. WALD [6] uses the ratio

$$u = \frac{p q'}{p' q}.$$

On the line $p + p' = 1$ δ can be expressed in terms of u:

(3.15)
$$u = \left(\frac{1+\delta}{1-\delta}\right)^2,$$

which is equivalent to

(3.16)
$$\begin{cases} \delta = 0 & \text{if} \quad u = 1 \\ \delta = \dfrac{u + 1 - 2\sqrt{u}}{u - 1} & \text{if} \quad u \neq 1. \end{cases}$$

Choosing four values for u with

(.317)
$$u_1 < u_2 < 1 < u_3 < u_4$$

one finds four values for δ such that

(3.18)
$$\delta_1 < \delta_2 < 0 < \delta_3 < \delta_4.$$

2. On the line $p + p' = 1$ the equality

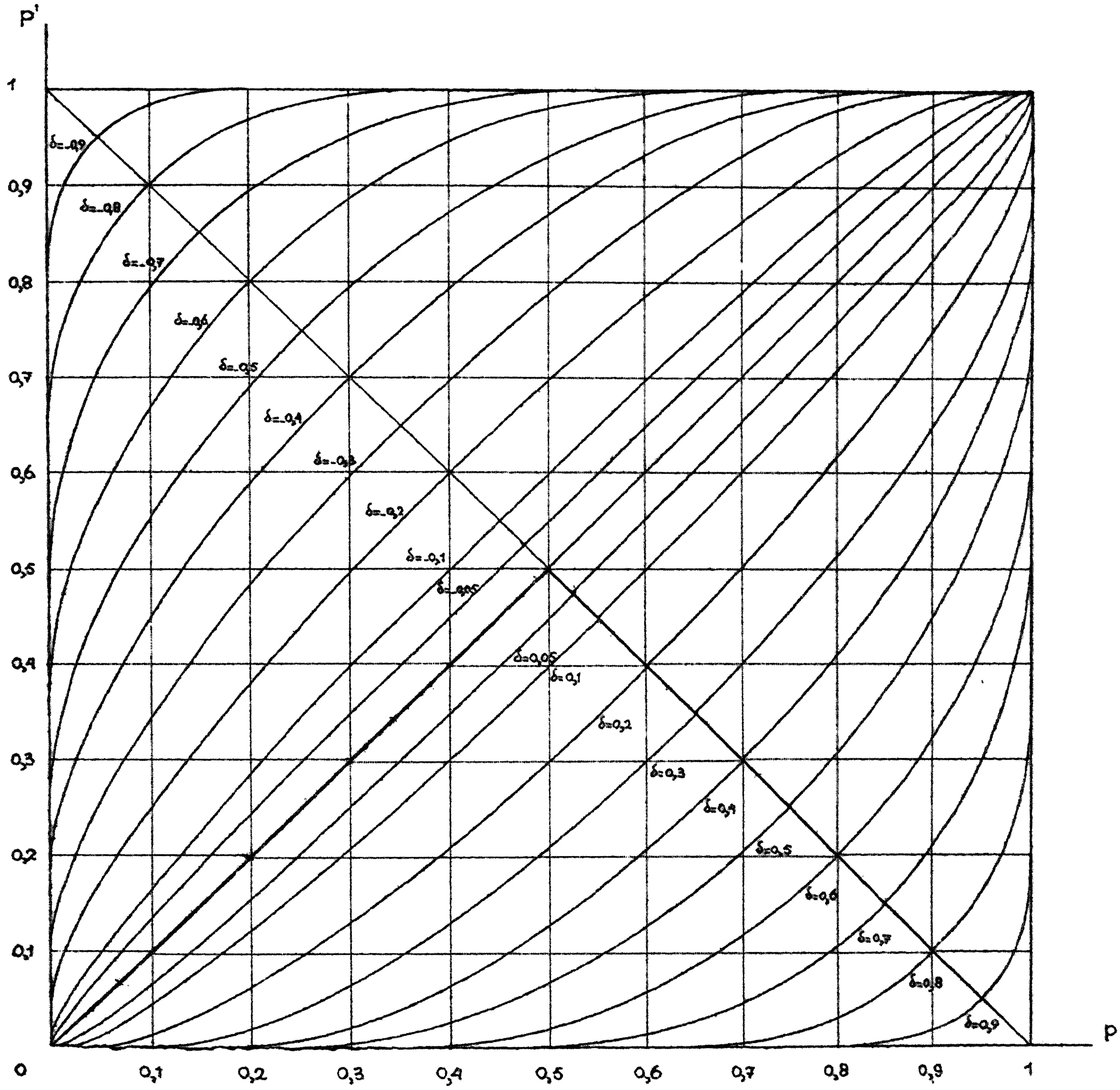
(3.19)
$$\delta = p - p'$$

holds.

Choosing four values for $p - p'$ one finds four values for δ. The four values of δ (or u respectively) must furthermore be chosen such that (3.7) holds.

Usually one will choose these values symmetrically, i.e. such that

Figure 2



Functional relationship between p and p' for several values of δ.

$$(3.20)\quad \begin{cases} \mu_4 = -\mu_1 \\ \mu_3 = -\mu_2 \end{cases}$$

which is equivalent to

$$(3.21)\quad \begin{cases} \delta_4 = -\delta_1 \\ \delta_3 = -\delta_2 \end{cases}$$

and to

$$(3.22)\quad u_1 u_4 = u_2 u_3 = 1$$

If (3.20) holds, (3.7) is equivalent to

$$(3.23)\quad \begin{cases} B \leq B' \\ A \leq A' \end{cases}$$

Remark

It is not necessary that p and p' are constants. We only need a constant δ .

References.

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